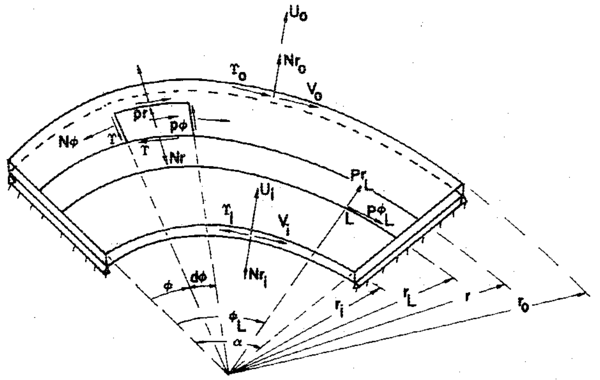

Fig. 1 Circular sector plate and its element.

normal and shear forces in terms of the initial parameters $U_{lm}, V_{lm}, Nr_{lm}, \Upsilon_{lm}$, and loads $Pr_{Lm}, P\phi_{Lm}$:

$$
\begin{bmatrix} U \\ V \\ r\,Nr/F \\ r\,\Upsilon/F \\ r\,N\phi/F \end{bmatrix}_{\rho m}
=
\underbrace{\begin{bmatrix} A_- & B_- & C_- & D_+ \\ F_- & B_+ & D_- & C_+ \\ E_+ & G_+ & A_+ & F_+ \\ G_- & E_- & H_- & H_+ \\ J_+ & J_- & I_+ & I_- \end{bmatrix}}_{T_{\rho l}}
\underbrace{\begin{bmatrix} \bar{C}_- & \bar{D}_+ \\ \bar{D}_- & \bar{C}_+ \\ \bar{A}_+ & \bar{F}_+ \\ \bar{H}_- & \bar{H}_+ \\ \bar{I}_+ & \bar{I}_- \end{bmatrix}}_{T_{\rho L}}
\begin{bmatrix} U_{lm} \\ V_{lm} \\ r_l\,Nr_{lm}/F \\ r_l\Upsilon_{lm}/F \\ \hline s r_L Pr_{Lm}/F \\ s r_L P\phi_{Lm}/F \end{bmatrix}
\tag{4}
$$

where matrices $T_{\rho l}$, $T_{\rho L}$, respectively, transport the mth coefficients of the initial parameters at the arc-line of radius r_l, and loads at the arc-line of radius r_L into deflections and normal and shear forces along the arc-line of radius r. Denoting $b_{\pm}=(\rho^{\alpha_m}\pm\rho^{-\alpha_m})/8$, $c_{\pm}=\rho\pm 1/\rho$, $\alpha_{m2}=1/(\alpha_m^2-1)$, $\nu_{\pm}=1\pm\nu$, $\nu_2=3-\nu$, $d_{\pm}=\alpha_m\nu_+c_{\pm}$, $d_m=\alpha_m d_-$, $x_-=\nu_-$, $x_+=2$, $y_-=0$, $y_+=\nu_+$, $z_-=\rho$, $z_+=c_+$, $e_{\pm}=d_m\mp 2x_{\pm}c_-$,

$$A_{\pm}=2(\nu_{\pm}\rho+\nu_{\mp}/\rho)b_+-d_-b_-$$

$$B_{\pm}=d_-b_{\mp}\pm 2x_{\pm}\rho b_{\pm}$$

$$C_{\pm}=\alpha_{m2}(\alpha_m\nu_2c_+b_-\pm e_{\pm}b_+)$$

$$D_{\pm}=\alpha_{m2}[(2\nu_2\rho^{\pm 1}\pm e_+)b_--\alpha_m\nu_2c_-b_+]$$

$$E_{\pm}=(2y_{\pm}c_-\mp d_m)b_++d_+b_-$$

$$F_{\pm}=(\pm 2\nu_-\rho^{\pm 1}-4c_-)b_-\pm d_-b_+$$

$$G_{\pm}=-d_-b_+\pm(d_m\mp 2\nu_+\rho^{\pm 1})b_-$$

$$H_{\pm}=2x_{\pm}b_{\pm}/\rho\pm d_-b_{\mp}$$

$$I_{\pm}=\pm 2(\nu_+\rho\mp x_{\mp}/\rho)b_{\pm}\pm d_-b_{\mp}$$

$$J_{\pm}=\pm(d_m+2\nu_+z_{\pm})b_{\pm}\pm(d_{\mp}+2\alpha_m\nu_+\rho)b_{\mp} \tag{5}$$

where when the double signs $\pm$, $\mp$ occur, the upper sign and symbols with the upper sign belong in one equation, and those corresponding to the lower sign in another equation.

The elements in the Pr and $P\phi$ columns of Eq. (4) are, respectively, obtained from those of Nr and Υ columns by simply replacing ρ with $\langle\bar{\rho}\rangle$, where it occurs explicitly and also in the notations $b_{\pm}$, $c_{\pm}$, $z_{\pm}$ used in Eqs. (5) (note that $\bar{\rho}=r/r_L$).

By choosing the initial parameters at the outer edge ($r_l=r_o$), the values of both ρ and $\bar{\rho}$ are limited to 1, thus avoiding large numbers in the numerical computation process.

Initial parameters approach enables complete definition of a behavior across the load discontinuities by a single equation. Knowing the parameters, the given equations enable evaluation of the distribution of in-plane deflections and normal and shear forces across the plate. The expressions will also readily facilitate solution for arbitrary conditions along the curved boundaries.

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Hydraulic Equivalent of Generalized One-Dimensional Flow

S. Boraas*
Bell Aerospace Textron, Buffalo, N. Y.

Introduction

THE similarity between the flow of a compressible gas and the flow of shallow water with a free surface is referred to as the hydraulic analogy. Since its development by Preiswerk,[1] the analog has been used successfully to study qualitatively both internal[2,3] and external[4-6] gas flowfields by flowing water in a rectangular channel. Bryant[7] has shown that this channel must have a constant rectangular shape if the analog is to be valid for a two-dimensional gas flow. Although he demonstrated this fact for an assumed isentropic gas, the same channel requirements exist if the flow is nonisentropic. Furthermore, his assumption of constant entropy leads to the conclusion that the water actually represents a fictitious gas with a specific heat ratio γ equal to 2.

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*High Energy Laser Technology. Member AIAA.

Following the approach used by Bryant, it can be shown that an analogy of a nonisentropic, one-dimensional gas flow is possible with a varying rectangular-shaped channel in which the water again represents a gas with a $\gamma = 2$. This type of an analogy has been used in recent studies of flows within chemical laser cavities.[8,9] These studies, which depict the nonisentropic flow as resulting solely from a heat release, suggested the development of the hydraulic equivalent of the generalized one-dimensional flow equations in which friction, internal drag, and mass flow change in the gas also contribute to the entropy increase. The purpose of this Note is to show the results of this development.

Development Results

The gas flow equations, Eqs. (6) and (7) of Ref. 9, were rederived to include not only the effect of a temperature change but also the effects of friction f, internal drag X, and mass flow change $d\dot{w}$. Using the influence coefficients from Ref. 10 for a gas with a constant specific heat ratio γ and molecular weight W, these equations are

$$\frac{dM^2}{M^2} = -\frac{2\left(1+\frac{\gamma-1}{2}M^2\right)}{(1-M^2)}\frac{dA}{A} + \frac{(1+\gamma M^2)\left(1+\frac{\gamma-1}{2}M^2\right)}{(1-M^2)} \times \frac{dT_0}{T_0} + \frac{2\gamma M^2\left(1+\frac{\gamma-1}{2}M^2\right)}{(1-M^2)}\left[2f\frac{dx}{D} + \frac{dX}{\gamma p A M^2}\right]$$
$$+ \frac{2\left(1+\frac{\gamma-1}{2}M^2\right)(1+(1-y)\gamma M^2)}{(1-M^2)}\frac{d\dot{w}}{\dot{w}} \tag{1}$$

and

$$\frac{dp}{p} = -\frac{\gamma M^2}{(1+\gamma M^2)}\left[\frac{dM^2}{M^2} + \frac{dA}{A} + \left(2f\frac{dx}{D} + \frac{dX}{\gamma p A M^2} - y\frac{d\dot{w}}{\dot{w}}\right)\right] \tag{2}$$

where D denotes the hydraulic diameter in the gas flow and y represents the ratio of the velocity component (in the direction of the main flow) of the mass of gas being added or subtracted to the velocity of the main flow. Similarly, the gas flow equations for a gas with a variable specific heat ratio and molecular weight (W) are

$$\frac{dM^2}{M^2} = -\frac{2\left(1+\frac{\gamma-1}{2}M^2\right)}{(1-M^2)}\frac{dA}{A} + \left(\frac{1+\gamma M^2}{1-M^2}\right)\left[\left(\frac{dQ-dW_X+dE}{C_pT}\right) - \frac{dW}{W}\right] + \frac{2\gamma M^2\left(1+\frac{\gamma-1}{2}M^2\right)}{(1-M^2)}\left[2f\frac{dx}{D} + \frac{dX}{\gamma p A M^2}\right]$$
$$+ \frac{2\left(1+\frac{\gamma-1}{2}M^2\right)(1+(1-y)\gamma M^2)}{(1-M^2)}\frac{d\dot{w}}{\dot{w}} - \frac{d\gamma}{\gamma} \tag{3}$$

and

$$\frac{dp}{p} = -\frac{\gamma M^2}{(1+\gamma M^2)}\left[\frac{dM^2}{M^2} + \frac{dA}{A} + \left(2f\frac{dx}{D} + \frac{dX}{\gamma p A M^2} - y\frac{d\dot{w}}{\dot{w}}\right) + \frac{d\gamma}{\gamma}\right] - \frac{\gamma M^2}{(1-M^2)}\frac{dW}{W} \tag{4}$$

where Q, W_X, and E denote, respectively, heat added, delivered work, and total energy.

The water flow equations, [Eqs. (8) and (9)] of Ref. 9, were revised to include only frictional effects on the basis that a mass change in the gas flow can be accounted for by mass change in the water flow and on the assumption that both frictional and internal drag effects in the gas can be represented by friction f' in the water. This assumption is certainly valid when the internal drag is the result of slowly moving particles and/or droplets, or the result of a body or gravity force acting on the gas, or a combination of these two. Although valid even when the internal drag is created by an immersed body, it would be desirable under such conditions to simulate the body in the water flow so as to more nearly achieve geometric similarity between the two flows. The revised water flow equations are

$$\frac{dF^2}{F^2} = -\frac{2(1+\tfrac{1}{2}a_1F^2)}{(1-a_4F^2)}\frac{dB}{B} + \frac{2(1+(2+a_2)F^2)}{(1-a_4F^2)}\frac{d\dot{w}}{\dot{w}} + \frac{6a_3F^2}{(1-a_4F^2)} \tag{5}$$

and

$$\frac{dH^2}{H^2} = -\frac{2F^2}{1+(2+a_7)F^2}\left[a_5\frac{dF^2}{F^2} + a_6\frac{dB}{B}\right] + \frac{4a_3F^2}{1+(2+a_7)F^2} \tag{6}$$

where

$$a_1 = 1 + \left(\frac{f'dx}{D'}\right)\cdot\left[2 + \left(\frac{D'}{2}\right)\left(\frac{(1+C)}{B} - \frac{3}{2H}\right)\right] \tag{7}$$

$$a_2 = \left(\frac{f'dx}{D'}\right)\cdot\left[1 + \frac{D'(1+C)}{4B}\right] \tag{8}$$

$$a_3 = f'dx/D' \tag{9}$$

$$a_4 = 1 + \left(\frac{f'dx}{D'}\right)\cdot\left[2 - \frac{D'(1+C)}{4B}\right] \tag{10}$$

$$a_5 = 1 + (f'dx/D') - \tfrac{1}{2}y' \tag{11}$$

$$a_6 = 1 + (f'dx/4H) - y' \tag{12}$$

$$a_7 = \left(\frac{f' \mathrm{d}x}{D'}\right) \cdot \left[1 + \frac{D'(1+C)}{4B}\right] - \frac{3}{2}y' \tag{13}$$

In these equations, D' denotes the hydraulic diameter at the entrance to the water channel, y' is the equivalent of y in that it is the ratio of the velocity component (in the direction of the main flow) of the mass of water being added or subtracted to the velocity of the main flow. The term C is a function of the change in the channel width in the flow direction and is defined as

$$C = \left[1 + \left(\frac{\mathrm{d}B}{\mathrm{d}x}\right)^2\right]^{1/2} \tag{14}$$

Equation (5) is the hydraulic equivalent of Eqs. (1) and (3), and a comparison of Eq. (5) with either Eq. (1) or (3) shows that the one-to-one correspondence between the A and B terms as demonstrated in Ref. 9 is now modified by the water friction through terms a_1 and a_4. A comparison of the remaining terms in Eq. (5) with those in Eqs. (1) and (3) shows that the combined effect of friction, internal drag, and changes in the gas composition, temperature, and mass flow rate can be simulated by a change in the water flow rate.

Equation (6) is the hydraulic equivalent of Eqs. (2) and (4), and a comparison of the former with either of the latter still shows a one-to-one correspondence between p and H^2. However, a similar relationship between M^2 and F^2 is altered through term a_5, which includes the effect of water friction and the velocity with which the water is added to or subtracted from the main channel flow.

An example of an application of the hydraulic equivalent is the hydraulic simulation of a confined, flowing, two-phase mixture injected with a combustible gas. Such a mixture will exhibit a mass change $\mathrm{d}\dot{w}$ due to the gas addition and changes in molecular weight $\mathrm{d}W$, specific heat ratio $\mathrm{d}\gamma$, and total energy $\mathrm{d}E$ due to combustion. Furthermore, the slower moving solid particles in the mixture will impose a drag $\mathrm{d}X$ on the gaseous flow and an uninsulated chamber wall could mean a heat loss $-\mathrm{d}Q$. Finally, this flowing, burning mixture might be used to deliver useful work $\mathrm{d}W_x$. In this case, Eqs. (3) and (4) express the combined effect of all these variations on the gaseous Mach number and static pressure. To find the equivalent water flow as expressed by these equations would require determining $\mathrm{d}p/p$ in Eq. (4) and then using Eqs. (5) and (6).

It is quite possible that sufficient information may not be available to simultaneously solve Eqs. (3) and (4) for $\mathrm{d}p/p$ in the preceding case. However, if $(\mathrm{d}p/p)$ for the flowing mixture is known from test data along with anticipated known values of $\mathrm{d}A/A$ and y, then the equivalent water flow can be determined. The procedure for determining this water flow is as follows:

1) Set $\mathrm{d}H^2/H^2 = \mathrm{d}p/p$ and $\mathrm{d}B/B = \mathrm{d}A/A$ in Eq. (6).

2) Calculate a_3, a_5, a_6, and a_7 assuming $f' = 0$ and $y' = y$.

3) Calculate $\mathrm{d}F^2/F^2$ from Eq. (6) and then the function $F = F(x)$ based upon the known boundary condition at the channel entrance $(x=0)$, $F = F_e = M_e$, as in Ref. 9.

4) With an average value of F over the simulation region, calculate the average water velocity and the value f' for a known water temperature.

5) Repeat steps 2-4 as often as necessary to obtain a final variation in F.

6) Calculate $\mathrm{d}\dot{w}/\dot{w}$ from Eq. (5) and then the function $\dot{w} = \dot{w}(x)$ based upon the known boundary condition at the channel entrance, $\dot{w} = \dot{w}_p$, where $\dot{w}_p = \dot{w}_p(M_e, B_e, H_g)$ denotes primary water flow rate as in Ref. 9.

In simulating the laser cavity flow of Ref. 9, the water flow was assumed frictionless. On this basis, the computed increase in water flow required to simulate the temperature increase was found to be 46.8%. It is interesting to note that when friction is considered and calculated using the expression of Ref. 11, the required increase in water flow was 36.4% assuming a water temperature of 20°C and normal injection $y = 0$. This does not change the results of Ref. 9, since the resultant static pressure and not the temperature increase was simulated there. However, it does emphasize the importance of water friction in an analog study, particularly when water must be added to simulate a nonisentropic gas flow.

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Heat and Mass Transfer in Close Proximity Impinging Two-Dimensional Laminar Jets

M. I. O. Ero*

Purdue University, West Lafayette, Ind.

Introduction

IMPINGING jets are of great practical interest in many industrial applications where they are commonly used as a means for attaining controlled heat and mass transfer rates on impingement surfaces. The study of jets impinging normally on a flat surface has been undertaken by many investigators. Gosman et al.[1] obtained solutions for the vorticity and temperature fields for the jet impinging at close proximity, but the transfer coefficients were not calculated. In the flow model of Gosman et al., the jet penetrated the solution domain with a fully developed Gaussian velocity profile of a free jet. In Scholtz and Trass,[2] the corresponding axisymmetric impingement flow was investigated in two steps; in the

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*Assistant Professor, Schoolo of Mechanical Engineering. Member AIAA.